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A general gauge graviton loop calculation

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Abstract. The one-loop graviton contribution to the graviton self-energy is calculated in a two-parameter gauge. The Slavnov and BRS identities are shown to be satisfied. A one-parameter family of gauges is also found in which the appropriate counterterm is part of a generally covariant object. Contrary to previous speculation, the spin-two and spin-zero parts of the counterterm can change sign. The significance of this result is discussed in detail.

1. Introduction

It might be thought that the renormalisation problem in quantum gravity could be resolved by adding matter in such a way that cancellations occur between the various loops contributing to a particular Green function, such as the graviton self-energy. However, it was shown some time ago that both in theory (Capper and Duff 1974, Deser and van Nieuwenhuizen 1974) and in practice (Capper *et al* 1974, Capper 1975a; for a review, see Nieuwenhuizen 1977) all the contributions to the infinities of a diagram seem to come in with the same sign. This led people to abandon any hope of obtaining a renormalisable theory of gravity and instead on-mass-shell finiteness was looked for (for a review, see van Nieuwenhuizen 1978). However, one problem has always been that, in a naive Feynman diagram approach, the counterterms required for the graviton loops are *not* part of a generally covariant object such as $\sqrt{-g}R^2$, etc, whereas those of the matter loops certainly are. In a previous paper (Capper and Namazie 1978) a one-parameter graviton gauge breaking term was used in an attempt to find a gauge in which the counterterms for the one-loop graviton self-energy were part of $\sqrt{-g}R^2$, $\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. The attempt was unsuccessful and the surprising feature was that this appeared to be because we insisted on the real world being four-dimensional. In this present paper we show that this failure was merely the result of an unfortunate choice of gauge-breaking term and graviton field parametrisation. Moreover, by employing a two-parameter gauge-breaking term, a one-parameter family of gauges can be obtained which all give rise to self-energy counterterms which are part of a generally covariant object. We also show that there are ranges of values of the gauge parameters for which the coefficients of the counterterms have the opposite sign to those of matter loops. Thus, by introducing appropriate matter interactions, it is possible to obtain a finite graviton self-energy. The possible significance of this result is discussed in the conclusion.

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2. An outline of the general gauge graviton loop calculation

Since similar calculations have been reported previously in the literature (Capper *et al* 1973, Capper and Namazie 1978), we merely give an outline here, mainly in order to fix our notation. In fact, such considerable use was made of the algebraic computer program SCHOONSCHIP (Strubbe 1974) that it would be impossible to describe the calculation in any detail.

We start from the action for pure gravity†

$$A=\frac{2}{K^2}\int \mathrm{d}^n x\sqrt{-g}R \tag{2.1}$$

where, in the spirit of dimensional regularisation ('t Hooft and Veltman 1972, Ashmore 1972, Bollini and Giambiagi 1972, for a review see Leibbrandt 1975), n is the dimension of space–time. In contrast to Capper and Namazie (1978), we define the graviton field $\phi_{\mu\nu}$ via

$$g_{\mu\nu}=\eta_{\mu\nu}+K\phi_{\mu\nu}. \tag{2.2}$$

The most general gaussian gauge-breaking term which is local and bilinear in the fields and derivatives is of the form:

$$\mathscr{L}_{\mathrm{B}}=(\alpha\,\partial_\mu\phi_{\mu\nu}+\beta\,\partial_\nu\phi_{\rho\rho})^2. \tag{2.3}$$

If we define projection operators (Delbourgo and Ramón Medrano 1976, Stelle 1977)

$$P^{(1)}_{\mu\nu\rho\sigma}=\tfrac{1}{2}(d_{\mu\rho}e_{\nu\sigma}+d_{\mu\sigma}e_{\nu\rho}+d_{\nu\rho}e_{\mu\sigma}+d_{\nu\sigma}e_{\mu\rho}) \tag{2.4}$$

$$P^{(2)}_{\mu\nu\rho\sigma}=\tfrac{1}{2}(d_{\mu\rho}d_{\nu\sigma}+d_{\mu\sigma}d_{\nu\rho})-\frac{1}{(n-1)}d_{\mu\nu}d_{\rho\sigma} \tag{2.5}$$

$$P^{\mathrm{o-s}}_{\mu\nu\rho\sigma}=\frac{1}{(n-1)}d_{\mu\nu}d_{\rho\sigma} \tag{2.6}$$

$$\mathrm{P}^{\mathrm{o-w}}_{\mu\nu\rho\sigma}=e_{\mu\nu}e_{\rho\sigma} \tag{2.7}$$

$$P^{\mathrm{o-sw}}_{\mu\nu\rho\sigma}=\frac{1}{\sqrt{n-1}}d_{\mu\nu}e_{\rho\sigma} \tag{2.8}$$

$$P^{\mathrm{o-ws}}_{\mu\nu\rho\sigma}=\frac{1}{\sqrt{n-1}}e_{\mu\nu}d_{\rho\sigma} \tag{2.9}$$

where

$$d_{\mu\nu}=\delta_{\mu\nu}-(p_\mu p_\nu/p^2) \tag{2.10}$$

$$e_{\mu\nu}=p_\mu p_\nu/p^2, \tag{2.11}$$

then the graviton propagator $G_{\alpha\beta,\mu\nu}$ is

$$\begin{aligned}G_{\alpha\beta,\mu\nu}=\frac{-\mathrm{i}}{(2\pi)^n(p^2+\mathrm{i}\epsilon)}\bigg[&\frac{1}{\alpha^2}P^{(1)}_{\alpha\beta\mu\nu}+P^{(2)}_{\alpha\beta\mu\nu}+\frac{1}{(2-n)}P^{\mathrm{o-s}}_{\alpha\beta\mu\nu}\\&+\left(\frac{2\beta^2(n-1)+(2-n)}{2(\alpha+\beta)^2(2-n)}\right)P^{\mathrm{o-w}}_{\alpha\beta\mu\nu}-\frac{\beta\sqrt{n-1}}{(\alpha+\beta)(2-n)}(P^{\mathrm{o-sw}}_{\alpha\beta\mu\nu}+P^{\mathrm{o-ws}}_{\alpha\beta\mu\nu})\bigg].\end{aligned} \tag{2.12}$$

† The notation used is specified in Capper and Namazie (1978). In particular, we use a $+---$ metric.

The ghost propagator $G_{\mu\nu}$ is given by

$$G_{\mu\nu}=\frac{1}{\mathrm{i}(2\pi)^n(p^2+\mathrm{i}\epsilon)}\left(\delta_{\mu\nu}-\frac{1}{2}\frac{p_\mu p_\nu}{p^2}\frac{(\alpha+2\beta)}{(\alpha+\beta)}\right) \tag{2.13}$$

and the ghost vertex (see figure 1) is

$$\begin{aligned}V_{\rho\sigma,\lambda,\mu}(k_1,k_2,k_3)\\ &=-\mathrm{i}(2\pi)^n[k_1\,.\,k_3\delta_{\rho\mu}\delta_{\sigma\lambda}+k_3^2\delta_{\rho\lambda}\delta_{\sigma\mu}+k_{1_\sigma}k_{3_\lambda}\delta_{\rho\mu}+k_{3_\sigma}k_{3_\lambda}\delta_{\rho\mu}+k_{1_\mu}k_{3_\rho}\delta_{\lambda\sigma}\\ &\quad+k_{1_\rho}k_{1_\mu}\delta_{\sigma\lambda}+(\beta/\alpha)(2k_{1_\lambda}k_{3_\sigma}\delta_{\rho\mu}+2k_{3_\lambda}k_{3_\sigma}\delta_{\rho\mu}+k_{1_\lambda}k_{1_\mu}\delta_{\rho\sigma}+k_{3_\lambda}k_{1_\mu}\delta_{\rho\sigma})],\end{aligned} \tag{2.14}$$

with implicit symmetrisation over ρ and σ. The three-graviton vertex was worked out by computer, and little purpose would be served by quoting it here. Suffice it to say that it appeared to agree with that of De Witt (1967).

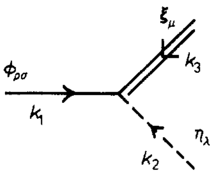

Figure 1. The graviton–fictitious particle vertex. The fictitious particles ξ_μ and η_λ have momentum labels k_3 and k_2 respectively. The graviton field $\phi_{\rho\sigma}$ has a momentum label k_1.

If we define $T_{\alpha\beta\alpha'\beta'}$ as the graviton self-energy shown in figure 2 and write it as

$$\begin{aligned}T_{\alpha\beta\alpha'\beta'}=p_\alpha p_\beta p_{\alpha'}p_{\beta'}T_1(p^2)+(p^2)^2\delta_{\alpha\beta}\delta_{\alpha'\beta'}T_2(p^2)\\ &+(p^2)^2(\delta_{\alpha\alpha'}\delta_{\beta'\beta}+\delta_{\alpha\beta'}\delta_{\alpha'\beta})T_3(p^2)+p^2(\delta_{\alpha\beta}p_{\alpha'}p_{\beta'}+\delta_{\alpha'\beta'}p_\alpha p_\beta)T_4(p^2)\\ &+p^2(\delta_{\alpha\alpha'}p_\beta p_{\beta'}+\delta_{\beta\alpha'}p_\alpha p_\beta+\delta_{\alpha\beta'}p_\beta p_{\alpha'}+\delta_{\beta\beta'}p_\alpha p_{\alpha'})T_5(p^2),\end{aligned} \tag{2.15}$$

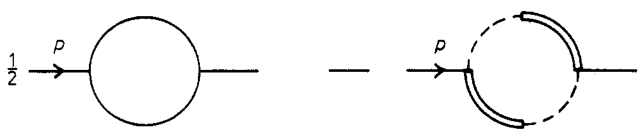

Figure 2. One-loop contributions to the graviton self-energy.

then we obtain for the pole terms†

$$\begin{aligned}T_1=[\tfrac{245}{8}+\tfrac{2265}{16}\alpha\beta+\tfrac{767}{4}\alpha\beta^3+\tfrac{435}{16}\alpha^2+\tfrac{1461}{8}\alpha^2\beta^2+\tfrac{1373}{16}\alpha^3\beta+\tfrac{257}{16}\alpha^4\\ &+30(\beta^4/\alpha^4)+120(\beta^3/\alpha^3)+180(\beta^2/\alpha^2)+120(\beta/\alpha)+\tfrac{915}{4}(\beta^3/\alpha)\\ &+\tfrac{2175}{8}\beta^2+\tfrac{169}{2}\beta^4+\tfrac{135}{2}(\beta^4/\alpha^2)]\bar{I}/(\alpha+\beta)^4\end{aligned} \tag{2.16}$$

† The amplitudes T_1 to T_5 were, in fact, evaluated for general n. The results are not quoted here since they are very complicated, but they were used to verify the n-dimensional Slavnov identities of § 3.

$$T_2=[\tfrac{735}{32}-\tfrac{725}{16}\alpha\beta+\tfrac{609}{4}\alpha\beta^3-\tfrac{115}{8}\alpha^2+\tfrac{1417}{8}\alpha^2\beta^2+\tfrac{1571}{16}\alpha^3\beta+\tfrac{673}{32}\alpha^4+\tfrac{45}{2}(\beta^4/\alpha^4)+90(\beta^3/\alpha^3)$$
$$+135(\beta^2/\alpha^2)+90(\beta/\alpha)-\tfrac{55}{4}(\beta^3/\alpha)-\tfrac{95}{2}\beta^2+\tfrac{113}{2}\beta^4]\bar{I}/(\alpha+\beta)^4 \tag{2.17}$$

$$T_3=[\tfrac{245}{64}+\tfrac{985}{64}\alpha\beta-\tfrac{289}{16}\alpha\beta^3+\tfrac{245}{64}\alpha^2-\tfrac{447}{32}\alpha^2\beta^2-\tfrac{11}{64}\alpha^3\beta+\tfrac{29}{16}\alpha^4+\tfrac{15}{4}(\beta^4/\alpha^4)$$
$$+15(\beta^3/\alpha^3)+\tfrac{45}{2}(\beta^2/\alpha^2)+15(\beta/\alpha)+\tfrac{315}{16}(\beta^3/\alpha)+\tfrac{805}{32}\beta^2-\tfrac{43}{8}\beta^4]\bar{I}/(\alpha+\beta)^4 \tag{2.18}$$

$$T_4=[-\tfrac{735}{32}-\tfrac{2095}{64}\alpha\beta-\tfrac{3041}{16}\alpha\beta^3-\tfrac{165}{64}\alpha^2-\tfrac{6203}{32}\alpha^2\beta^2-\tfrac{5899}{64}\alpha^3\beta-\tfrac{1071}{64}\alpha^4-\tfrac{45}{2}(\beta^4/\alpha^4)$$
$$-90(\beta^3/\alpha^3)-135(\beta^2/\alpha)-90(\beta/\alpha)-\tfrac{1045}{16}(\beta^3/\alpha)-\tfrac{2785}{32}\beta^2-\tfrac{607}{8}\beta^4$$
$$-\tfrac{225}{8}(\beta^4/\alpha^2)]\bar{I}/(\alpha+\beta)^4 \tag{2.19}$$

$$T_5=-T_3 \tag{2.20}$$

where

$$\bar{I}=\frac{\mathrm{i}\pi^2K^2}{60(\omega-2)} \qquad (\text{with } 2\omega=n). \tag{2.21}$$

The evaluation of the integrals involved in this calculation is explained in appendix 1. The validity of equations (2.16)–(2.20) can be checked by substitution in the following Ward identities, obtained as in Capper *et al* (1973):

$$T_3+T_5=0 \tag{2.22}$$

$$T_1+T_2+2T_4+2T_5=0. \tag{2.23}$$

3. The Slavnov and BRS identities

The relevant Slavnov identities may be derived as Capper and Ramón Medrano (1974) and are contained in the equation

$$\frac{2\alpha}{K}\langle T\phi_{\mu\nu}(z)(\alpha\phi_{\lambda\rho,\lambda}(y)+\beta\phi_{\lambda\lambda,\rho}(y))\rangle=\langle T(A_{\mu\nu\lambda}(z)\xi_\lambda(z)\eta_\rho(y))\rangle \tag{3.1}$$

where

$$A_{\mu\nu\lambda}=K^{-1}(\delta_{\nu\lambda}\partial_\mu+\delta_{\mu\lambda}\partial_\nu)+(\phi_{\lambda\nu}\partial_\mu+\phi_{\lambda\mu}\partial_\nu+\partial_\lambda\phi_{\mu\nu}) \tag{3.2}$$

and ξ_λ and η_ρ are the fictitious particle fields†. The above identity was verified using the T_i's of equations (2.16)–(2.20) and, moreover, it was also verified for the n-dimensional versions of these equations. This is an extremely stringent check on the reliability of our calculations, involving as it does the independent evaluation of three sets of diagrams (see figures 2, 3 and 4) which are functions of the three independent parameters α, β, n.

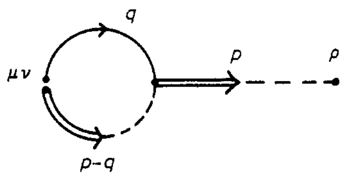

Figure 3. The lowest-order one-loop contribution to $\langle T(A_{\mu\nu\lambda}(z)\xi_\lambda(z)\eta_\rho(y))\rangle$.

† See figure 1 and Capper *et al* (1973), Capper and Ramón Medrano (1974) and Capper and Namazie (1978).

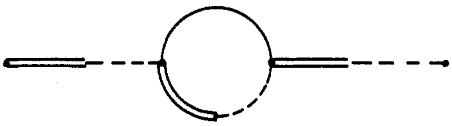

Figure 4. The one-loop contribution to the fictitious particle self-energy.

However, it is actually more illuminating to consider the BRS identities and it was originally hoped that these would enable us to pick out a combination of gauge parameters which would render the counterterm for figure 2 part of a generally covariant object.

Let us define, as in Delbourgo and Ramón Medrano (1976)†,

$$T_{\alpha\beta\alpha'\beta'} = [2P^{(2)}_{\alpha\beta\alpha'\beta'}E_1 + 2P^{(1)}_{\alpha\beta\alpha'\beta'}E_2 + P^{\mathrm{o-w}}_{\alpha\beta\alpha'\beta'}E_3 + 3P^{\mathrm{o-s}}_{\alpha\beta\alpha'\beta'}E_4 + \sqrt{3}(P^{\mathrm{o-sw}}_{\alpha\beta\alpha'\beta'} + P^{\mathrm{o-ws}}_{\alpha\beta\alpha'\beta'})E_5](p^2)^2 \tag{3.3}$$

where the projection operators $P^{(i)}$ are now in four dimensions since we are only interested in the pole terms. Then we find that

$$E_1 = T_3 \tag{3.4}$$

$$E_2 = T_3 + T_5 \tag{3.5}$$

$$E_3 = T_1 + T_2 + 2T_3 + 2T_4 + 4T_5 \tag{3.6}$$

$$E_4 = T_2 + \tfrac{2}{3}T_3 \tag{3.7}$$

$$E_5 = T_2 + T_4. \tag{3.8}$$

Defining

$$y = 2\beta/\alpha \tag{3.9}$$

and

$$x = \alpha^2 \tag{3.10}$$

we obtain

$$E_1 = \left(\frac{5}{2x^2}(\tfrac{49}{32} + 3y + \tfrac{9}{4}y^2 + \tfrac{3}{4}y^3 + \tfrac{3}{32}y^4) + \frac{5}{64x}(49 + \tfrac{197}{2}y + \tfrac{161}{2}y^2 + \tfrac{63}{2}y^3 + \tfrac{9}{2}y^4) + \tfrac{1}{16}(29 - \tfrac{11}{8}y - \tfrac{447}{8}y^2 - \tfrac{289}{8}y^3 - \tfrac{43}{8}y^4)\right)\frac{\bar{I}}{(1+\frac{1}{2}y)^4} \tag{3.11}$$

$$E_2 = 0 \tag{3.12}$$

$$E_3 = 0 \tag{3.13}$$

$$E_4 = \left(\frac{25}{x^2}(\tfrac{49}{48} + 2y + \tfrac{3}{2}y^2 + \tfrac{1}{2}y^3 + \tfrac{1}{16}y^4) + \frac{5}{32x}(-\tfrac{227}{3} - \tfrac{673}{6}y - \tfrac{295}{6}y^2 - \tfrac{1}{2}y^3 + \tfrac{3}{2}y^4) + \tfrac{5}{32}(\tfrac{427}{3} + \tfrac{1883}{6}y + \tfrac{537}{2}y^2 + \tfrac{673}{6}y^3 + \tfrac{127}{6}y^4)\right)\frac{\bar{I}}{(1+\frac{1}{2}y)^4} \tag{3.14}$$

† We keep with the notation of Delbourgo and Ramón Medrano (1976), although E_1 is the coefficient of the spin-two projection operator and E_2 the coefficient of the spin-one projection operator!

$$E_5=\frac{5}{64}\left((55+11y-59y^2-31y^3)+\frac{1}{x}(-217-391y-235y^2-45y^3)\right)\frac{\bar{I}}{(1+\frac{1}{2}y)^3}. \quad (3.15)$$

The condition that the counterterms for figure 2 are contained in†

$$\Delta\mathcal{L}=\frac{-\sqrt{-g}}{60(\omega-2)(4\pi)^2}[aR^2+b(R_{\mu\nu}R^{\mu\nu}-\tfrac{1}{2}R^2)] \quad (3.16)$$

is that $E_5=0$, which gives a condition for x‡:

$$x=\frac{217+391y+235y^2+45y^3}{55+11y-59y^2-31y^3}. \quad (3.17)$$

With this condition imposed, we obtain

$$a=\tfrac{1}{2}E_4 \qquad b=4E_1. \quad (3.18)$$

The functions E_1 and E_4 are plotted in figures 5 and 6 and displayed in detail in appendix 2. For comparison, the corresponding values for the photon are

$$a=0 \qquad b=12 \quad (3.19)$$

and for a massless scalar field

$$a=\tfrac{5}{6} \qquad b=1. \quad (3.20)$$

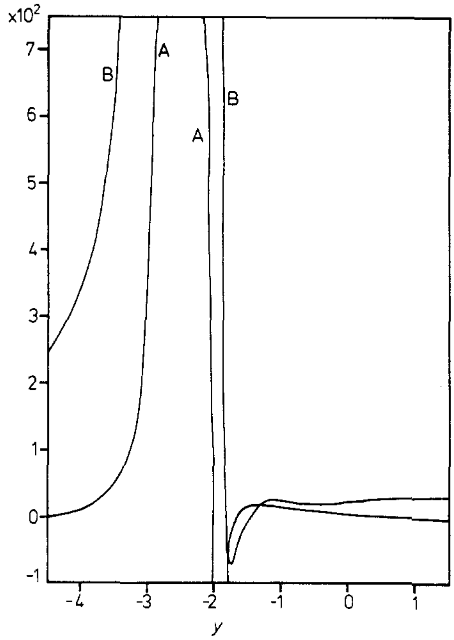

Figure 5. A graph showing how (A) $E_1/\bar{I}$ and (B) $E_4/\bar{I}$ vary with the gauge parameter y, subject to the condition $E_5=0$.

† If, in the spirit of Capper (1979), we look for n-dimensional counterterms, then equation (3.16) could involve $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$; however, due to the fact that the bilinear part of the Gauss–Bonnet formula directly generalises to n dimensions, the coefficient of such a term cannot be found from a self-energy calculation.
‡ Note that we do not obtain generally covariant counterterms for $\beta=0$, $\alpha=\infty$, i.e., for $y=0$ we get $x\neq\infty$ (cf, however, § 11 of Stelle 1977).

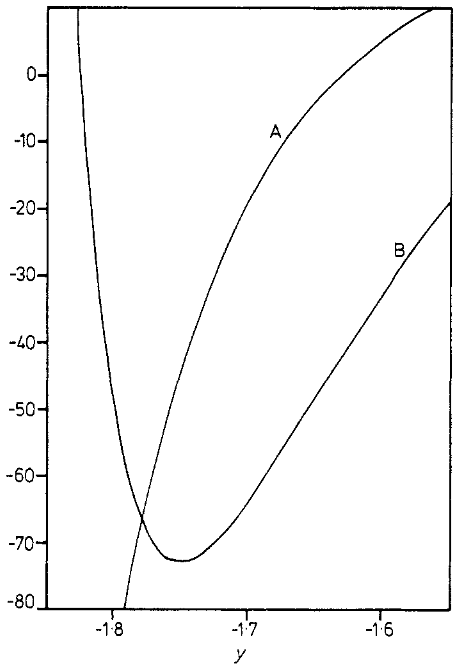

Figure 6. A graph showing in more detail than figure 5 the region in which (A) $E_1/\bar{I}$ and (B) $E_4/\bar{I}$ go negative.

For a non-minimally coupled scalar field, the coefficient a varies and can be made zero, but not negative. The coefficient b is unaffected by non-minimal couplings.

To verify the BRS identities we follow Debourgo and Ramón Medrano (1976). Defining $\hat{W}_{\mu\nu\rho}$ and $\hat{\mathscr{F}}_{\nu\rho}$ as the amputated parts of the amplitudes shown in figures 3 and 4 respectively, we may write

$$\hat{W}_{\mu\nu\rho} = p_\rho d_{\mu\nu} F + p_\rho e_{\mu\nu} G + d_{\mu\rho} p_\nu + d_{\nu\rho} p_\mu H \tag{3.21}$$

and

$$\hat{\mathscr{F}}_{\nu\rho} = d_{\nu\rho} A + e_{\nu\rho} B. \tag{3.22}$$

Computer calculations show that the infinite parts of A, B, F, G, H are given by†

$$A = \frac{5}{64(1+\frac{1}{2}y)^3}\left[y^3\left(97-\frac{29}{x}\right)+y^2\left(481-\frac{127}{x}\right)+y\left(847-\frac{175}{x}\right)+\left(471-\frac{49}{x}\right)\right]\bar{I} \tag{3.23}$$

$$B = \frac{15}{64(1+\frac{1}{2}y)^3}\left[y^4\left(8-\frac{40}{x}\right)+y^3\left(143-\frac{235}{x}\right)\right.$$
$$\left.+y^2\left(479-\frac{485}{x}\right)+y\left(593-\frac{413}{x}\right)+\left(241-\frac{111}{x}\right)\right]\bar{I} \tag{3.24}$$

$$F = \frac{5}{64(1+\frac{1}{2}y)^3}\left[y^3\left(31+\frac{45}{x}\right)+y^3\left(59+\frac{235}{x}\right)+y\left(-11+\frac{391}{x}\right)+\left(-55+\frac{217}{x}\right)\bar{I}\right. \tag{3.25}$$

† Expressions for F, G, H, A, B for general n were found implicitly when the Slavnov identity of equation (3.1) was verified.

$$G=\frac{15}{64(1+\frac{1}{2}y)^3}\left[y^3\left(-47+\frac{35}{x}\right)+y^2\left(-251+\frac{165}{x}\right)+y\left(-445+\frac{249}{x}\right)+\left(-241+\frac{111}{x}\right)\right]\bar{I} \tag{3.26}$$

$$H=\frac{5}{64(1+\frac{1}{2}y)^3}\left[y^3\left(-97+\frac{29}{x}\right)+y^2\left(-481+\frac{127}{x}\right)+y\left(-847+\frac{175}{x}\right)+\left(-471+\frac{49}{x}\right)\right]\bar{I}. \tag{3.27}$$

Using the results of Zinn-Justin (1974) and Delbourgo and Ramón Medrano (1976), the general gauge BRS identities are

$$E_2=0 \tag{3.28}$$

$$E_3=0 \tag{3.29}$$

$$F=-E_5 \tag{3.30}$$

and the ghost equation of motion gives

$$A=-H \tag{3.31}$$

$$B=-\tfrac{3}{2}yF-(1+\tfrac{1}{2}y)G. \tag{3.32}$$

It can be verified that equations (3.28)–(3.32) are indeed satisfied by equations (3.23)–(3.27). As can be seen, the condition for generally covariant counterterms is that

$$E_5=0, \tag{3.33}$$

which simply gives us an equation for x in terms of y, i.e., equation (3.17). Unfortunately, the BRS identities do not seem to give any motivation for choosing any particular combination of x and y which satisfy this equation.

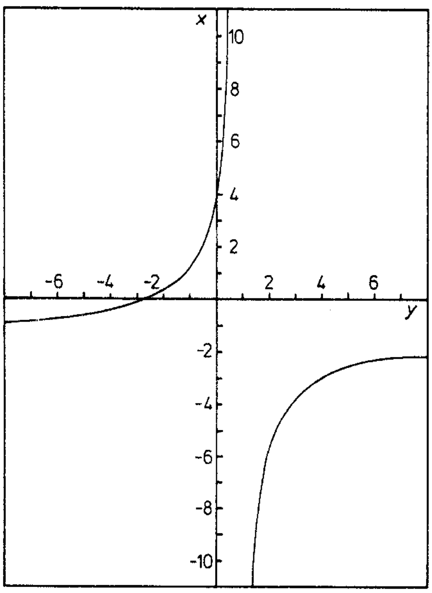

Figure 7. A graph showing how the gauge parameters x and y vary, when constrained by the condition $E_5=0$.

A graph showing values of x for real values of y is given in figure 7. Various simple values of x and y are given in table 1. The present calculation does not seem to indicate that any special significance should be attached to these values†. In fact, the gauge $y=-2$, $x=\frac{1}{3}$ must be excluded since it is singular.

Table 1.

x	y
1	-1
$\frac{1}{3}$	-2
$-\frac{4}{9}$	-4

4. Some other one-loop diagrams

One is of course really interest in many diagrams other than just the graviton self-energy. Two of these, which are readily evaluated using the results outlined in appendix 1, are the graviton contributions to the photon and scalar self-energies shown respectively in figures 8 and 9. The infinite part of the photon self-energy $\pi_{\mu\nu}(p^2)$ is given by

$$\pi_{\mu\nu}(p^2)=\left(\frac{p_\mu p_\nu}{p^2}-\delta_{\mu\nu}\right)\frac{5(p^2)^2}{(1+\frac{1}{2}y)^2}\left((1+2y)^2-\frac{3}{x}\right)\bar{I}. \tag{4.1}$$

Figure 8. The one-loop graviton contribution to the photon self-energy.

Figure 9. The one-loop graviton contribution to the scalar self-energy.

For the infinite part of the scalar self-energy $\pi(p^2)$ we obtain

$$\pi(p^2)=\frac{45(1+y)}{8(1+\frac{1}{2}y)^2}(p^2)^2\left(\frac{(3+y)}{x}-(5+y)\right)\bar{I}. \tag{4.2}$$

It is noteworthy that values of x and y can be found for which either or even both of the expressions in equations (4.1) and (4.2) in fact vanish.

† A check was made on the computer results by substituting the values $y=-\frac{7}{4}$, $x=\frac{721}{1357}$ in equations (2.16)–(2.20) and checking by hand that these led to the results implied by figure 6.

5. Conclusion

The original motivation for this work was to investigate the rather strange result of Capper and Namazie (1978) which seems to indicate that there is something special about four-dimensional space–time which makes it impossible to choose a gauge parameter such that the self-energy counterterms are generally covariant. This is why Feynman diagram techniques are employed rather than the background field method. A related calculation has in fact been performed by Kallosh *et al* (1978) using the background field technique. They found that it was even possible to choose gauge parameters which made all the off-mass-shell one-loop graviton counterterms vanish (albeit for complex gauge parameters). As shown in appendix 2, our approach does not lead to this conclusion. However, there is not necessarily an inconsistency since the two techniques are somewhat different and may not lead to the same off-mass-shell amplitudes. Another possibility is that we could introduce a weight parameter for the graviton field which might lead to the vanishing of off-mass-shell counterterms in our approach. It is impossible to say whether or not this would remove all one-loop divergences without a detailed analysis of the BRS identities. In view of the results of Capper (1979) this seems unlikely.

We have also demonstrated in this paper that for certain ranges of values of the gauge parameters, the coefficients of the spin-zero and spin-two parts of the counterterms (i.e., E_4 and E_1 respectively) can change sign. At first it might be thought that this contradicts the theorems of Capper and Duff (1974) and Deser and van Nieuwenhuizen (1974), as well as our experience of other field theories such as quantum electrodynamics. Apparently the presence of graviton, as opposed to matter, loops renders these theorems inapplicable. The reason for this is explained in detail in appendix 3.

Since it seems possible to choose a gauge in which the graviton contributions to the graviton self-energy counterterm are opposite to those of matter, it is worthwhile speculating on the possibility of an exact cancellation. If gravity is only allowed to couple minimally to matter, then the matter counterterms (corresponding to E_1 and E_4) can only vary in discrete steps. Introducing non-minimal couplings would enable an exact cancellation to occur. However, since we have a one-parameter family of gauges for which E_5 vanishes (see figure 6) the number of particles coupling to gravity would vary with the gauge! This is, or course, unreasonable and a more likely possibility is that a consideration of other one-loop graviton diagrams fixes the gauge uniquely. Indeed it may even turn out to be impossible to find a gauge in which the counterterms for all the infinity of one-loop graviton diagrams are of the same form as those due to matter loops. Even if this is not the case, then to get cancellations for all one-loop diagrams we are faced with the almost impossible task of choosing the correct numbers of each type of particle together with their various non-minimal interactions. A more promising approach might be to hope that a particular supergravity theory is the 'correct' one and to use supergraph techniques (Capper 1975b, Capper and Leibbrandt 1975) to demonstrate that the theory is finite, if indeed such a theory exists.

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Appendix 1. The evaluation of one-loop integrals

We present some results for one-loop massless integrals which are more powerful than those given by Capper *et al* (1973) and Capper and Namazie (1978).

First we define†

$$\int \frac{\mathrm{d}q^{2\omega}(p\,.\,q)^j}{(q^n)^n[(p-q)^2]^m} = I, (n, m, j) \tag{A.1}$$

$$\int \frac{\mathrm{d}q^{2\omega}(p\,.\,q)^j q_{\alpha_1}}{(q^n)^n[(p-q)^2]^m} = p_{\alpha_1} I_2(n, m, j) \tag{A.2}$$

$$\int \frac{\mathrm{d}q^{2\omega}(p\,.\,q)^j q_{\alpha_1} q_{\alpha_2}}{(q^2)[(p-q)^2]^m} = \delta_{\alpha_1\alpha_2} I_3(n, m, j) + p_{\alpha_1} p_{\alpha_2} I_4(n, m, j) \tag{A.3}$$

$$\begin{aligned} &\int \frac{\mathrm{d}q^{2\omega}(p\,.\,q)^j q_{\alpha_1} q_{\alpha_2} q_{\alpha_3}}{(q^2)^n[(p-q)^2]^m} \\ &\qquad = p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} I_5(n, m, j) + (\delta_{\alpha_1\alpha_2} p_{\alpha_3} + \delta_{\alpha_1\alpha_3} p_{\alpha_2} + \delta_{\alpha_2\alpha_3} p_{\alpha_1}) I_6(n, m, j) \end{aligned} \tag{A.4}$$

$$\begin{aligned} &\int \frac{\mathrm{d}q^{2\omega}(p\,.\,q)^j q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} q_{\alpha_4}}{(q^2)^n[(p-q)^2]^m} \\ &\qquad = p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} p_{\alpha_4} I_7(n, m, j) + (\delta_{\alpha_3\alpha_4} p_{\alpha_1} p_{\alpha_2} + \delta_{\alpha_2\alpha_4} p_{\alpha_1} p_{\alpha_3} + \delta_{\alpha_2\alpha_3} p_{\alpha_1} p_{\alpha_4} \\ &\qquad + \delta_{\alpha_1\alpha_4} p_{\alpha_2} p_{\alpha_3} + \delta_{\alpha_1\alpha_3} p_{\alpha_2} p_{\alpha_4} + \delta_{\alpha_1\alpha_2} p_{\alpha_3} p_{\alpha_4}) I_8(n, m, j) \\ &\qquad + (\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3} + \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} + \delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}) I_9(n, m, j). \end{aligned} \tag{A.5}$$

Unfortunately there are over 500 $I_K(n, m, j)$'s to evaluate even if we restrict ourselves to those integrals arising from the interactions considered in this paper! But we can derive the following identities:

$$I_K(n, 2, j) = p^2 I_K(n, j) - 2I_K(n, j+1) + I_K(n-1, j) \tag{A.6}$$

$$\begin{aligned} I_K(n, 1, j) = (p^2)^2 I_K(n, j) - 4p^2 I_K(n, j+1) - 4I_K(n-1, j+1) \\ + 2p^2 I_K(n-1, j) + 4I_K(n, j+2) + I_K(n-2, j) \end{aligned} \tag{A.7}$$

where

$$I_K(n, 3, j) = I_K(n, j). \tag{A.8}$$

Equations (A.6) and (A.7) result from multiplying the integrands of equations (A.1)–(A.4) by the appropriate powers of $(p-q)^2/(p-q)^2$ to render all the denominators of the form $[(p-q)^2]^3$. Use can then be made of the further identities

$$I_1(n, j) = I(n, j) \tag{A.9}$$

$$I_2(n, j) = I(n, j+1)/p^2 \tag{A.10}$$

$$I_3(n, j) = [I(n-1, j) - (p^2)^{-1} I(n, j+2)]/(2\omega - 1) \tag{A.11}$$

$$I_4(n, j) = [2\omega(p^2)^{-2} I(n, j+2) - (p^2)^{-1} I(n-1, j)]/(2\omega - 1) \tag{A.12}$$

$$I_5(n, j) = [2(\omega+1)(p^2)^{-3} I(n, j+3) - 3(p^2)^{-2} I(n-1, j+1)]/(2\omega - 1) \tag{A.13}$$

$$I_6(n, j) = [(p^2)^{-1} I(n-1, j+1) - (p^2)^{-2} I(n, j+3)]/(2\omega - 1) \tag{A.14}$$

† The integrals (A.1)–(A.5) are in Minkowski space, but for convenience we have omitted all factors of $i\epsilon$.

$$I_7(n, j) = [(p^2)^{-4}(4\omega^2 + 12\omega + 8)I(n, j+4)$$
$$-(p^2)^{-3}(12\omega + 12)I(n-1, j+2) + 3(p)^{-2}I(n-2, j)]/(4\omega^2 - 1) \tag{A.15}$$
$$I_8(n, j) = [(p^2)^{-2}(2\omega + 3)I(n-1, j+2) - (p^2)^{-3}2(\omega + 1)I(n, j+4)$$
$$-(p^2)^{-1}I(n-2, j)]/(4\omega^2 - 1) \tag{A.16}$$
$$I_9(n, j) = [-2(p^2)^{-1}I(n-1, j+2) + (p^2)^{-2}I(n, j+4) + I(n-2, j)]/(4\omega^2 - 1). \tag{A.17}$$

Equation (A.9)–(A.17) can be obtained by various combinations of tracing over Lorentz indices and taking scalar products with various numbers of p_{α_i}'s. We are now left with 28 integrals to evaluate. This can be done using the technique outlined by Capper *et al* (1973) and we obtain

$$I(3, 8) = \tfrac{1}{64}(p^2)^2(4\omega^4 - 52\omega^3 + 127\omega^2 + 161\omega + 48)I \tag{A.18}$$
$$I(2, 8) = \tfrac{1}{64}(p^2)^3(-4\omega^3 + 16\omega^2 + 65\omega + 48)I \tag{A.19}$$
$$I(1, 8) = \tfrac{1}{64}(p^2)^4(\omega + 2)(2\omega + 5)I \tag{A.20}$$
$$I(3, 7) = \tfrac{1}{32}p^2(4\omega^4 - 52\omega^3 + 155\omega^2 + 7\omega + 6)I \tag{A.21}$$
$$I(2, 7) = \tfrac{1}{16}(p^2)^2(\omega + 1)(-2\omega^2 + 11\omega + 9)I \tag{A.22}$$
$$I(1, 7) = \tfrac{1}{32}(p^2)^3(\omega + 2)(2\omega + 3)I \tag{A.23}$$
$$I(3, 6) = \tfrac{1}{16}(4\omega^4 - 52\omega^3 + 179\omega^2 - 125\omega + 30)I \tag{A.24}$$
$$I(2, 6) = \tfrac{1}{16}p^2(2\omega + 1)(-2\omega^2 + 11\omega + 3)I \tag{A.25}$$
$$I(1, 6) = \tfrac{1}{16}p^2(\omega + 1)(2\omega + 3)I \tag{A.26}$$
$$I(3, 5) = \tfrac{1}{8}(p^2)^{-1}(4\omega^4 - 52\omega^3 + 199\omega^2 - 235\omega + 90)I \tag{A.27}$$
$$I(2, 5) = \tfrac{1}{4}\omega(-2\omega^2 + 11\omega - 2)I \tag{A.28}$$
$$I(1, 5) = \tfrac{1}{8}p^2(2\omega + 1)(\omega + 1)I \tag{A.29}$$
$$I(3, 4) = \tfrac{1}{4}(p^2)^{-2}(4\omega^4 - 52\omega^3 + 215\omega^2 - 323\omega + 162)I \tag{A.30}$$
$$I(2, 4) = \tfrac{1}{4}(p^2)^{-1}(-4\omega^3 + 24\omega^2 - 23\omega + 6)I \tag{A.31}$$
$$I(1, 4) = \tfrac{1}{4}\omega(2\omega + 1)I \tag{A.32}$$
$$I(3, 3) = \tfrac{1}{2}(p^2)^{-3}(2\omega - 3)(4 - \omega)(-2\omega^2 + 15\omega - 19)I \tag{A.33}$$
$$I(2, 3) = (p^2)^{-2}(\omega - 1)(-2\omega^2 + 11\omega - 9)I \tag{A.34}$$
$$I(1, 3) = (p^2)^{-1}\tfrac{1}{2}\omega(2\omega - 1)I \tag{A.35}$$
$$I(3, 2) = (p^2)^{-4}(4 - \omega)(2\omega - 3)(-2\omega^2 + 15\omega - 23)I \tag{A.36}$$
$$I(2, 2) = (p^2)^{-3}(2\omega - 3)(-2\omega^2 + 11\omega - 11)I \tag{A.37}$$
$$I(1, 2) = (p^2)^{-2}(2\omega - 1)(\omega - 1)(\omega - 1)I \tag{A.38}$$
$$I(3, 1) = (p^2)^{-5}2(5 - \omega)(4 - \omega)(2\omega - 3)(2\omega - 5)I \tag{A.39}$$
$$I(2, 1) = (p^2)^{-4}2(4 - \omega)(2\omega - 3)(2\omega - 4)I \tag{A.40}$$
$$I(1, 1) = (p^2)^{-3}2(\omega - 1)(2\omega - 3)I \tag{A.41}$$
$$I(3, 0) = (p^2)^{-6}4(5 - \omega)(4 - \omega)(2\omega - 3)(2\omega - 5)I \tag{A.42}$$

$$I(2,0)=(p^2)^{-5}4(4-\omega)(2\omega-3)(2\omega-5)I \tag{A.43}$$

$$I(1,0)=(p^2)^{-4}2(2\omega-3)(2\omega-4)I \tag{A.44}$$

where

$$I=\frac{(p^2)^2}{4}\int\frac{\mathrm{d}^{2\omega}q}{q^2(p-q)^2} \tag{A.45}$$

and

$$\int\frac{\mathrm{d}^{2\omega}q}{q^2(p-q)^2}=\mathrm{i}\pi^{\omega}\Gamma(2-\omega)\Gamma(\omega-1)\Gamma(\omega-1)(\Gamma(2\omega-2))^{-1}(-p^2-\mathrm{i}\epsilon)^{\omega-2}. \tag{A.46}$$

No doubt there are more identities which could reduce the number of integrals to be evaluated even further; however, it is safer to leave some redundancy to enable the results to be checked. In fact, a computer program was used to generate all the relevant $I_K(n, m, j)$'s in the form of a SCHOONSCHIP program. This was then checked by using it to evaluate various massless tadpoles which, in the context of dimensional regulation, are formally zero (Capper and Leibbrandt 1974).

Appendix 2

We list here the result of substituting equation (3.17) in equations (3.11) and (3.14). These functions $E_1(y)$, $E_4(y)$ are plotted in figures 5 and 6.

$$E_1(y)=\Big(\tfrac{2281881}{64}+\tfrac{1486123}{16}y+\tfrac{2452789}{64}y^2-\tfrac{4262911}{32}y^3-\tfrac{14442179}{64}y^4-\tfrac{1299285}{8}y^5 -\tfrac{3970445}{64}y^6-\tfrac{389235}{32}y^7-\tfrac{30255}{32}y^8\Big)\frac{4\bar{I}}{(\frac{1}{2}y+1)^2(217+391y+235y^2+45y^3)^2} \tag{A.47}$$

$$E_4(y)=\Big(\tfrac{3933335}{16}+\tfrac{14370455}{12}y+\tfrac{124531745}{48}y^2+\tfrac{78381685}{24}y^3+\tfrac{125558585}{48}y^4+\tfrac{2735325}{2}y^5 +\tfrac{21935495}{48}y^6+\tfrac{715675}{8}y^7+\tfrac{62975}{8}y^8\Big)\frac{4\bar{I}}{(\frac{1}{2}y+1)^2(217+391y+235y^2+45y^3)^2}. \tag{A.48}$$

We note that the approximate roots of

$$E_1=0 \tag{A.49}$$

are $y=-4{\cdot}521$, $-2{\cdot}042$, $0{\cdot}6723$, $-1{\cdot}329\pm0{\cdot}4026\mathrm{i}$, $-1{\cdot}342\pm0{\cdot}3594\mathrm{i}$ and $-1{\cdot}632$. The approximate roots of

$$E_4=0 \tag{A.50}$$

are $y=-1{\cdot}475\pm1{\cdot}143\mathrm{i}$, $-1{\cdot}207\pm0{\cdot}4938\mathrm{i}$, $-1{\cdot}357\pm0{\cdot}3681\mathrm{i}$, $-1{\cdot}824$ and $-1{\cdot}463$. As can be seen, there are no simultaneous roots of equations (A.49) and (A.50). Thus, in contrast to the approach of Kallosh *et al* (1978), it is impossible to find a gauge in which all the counterterms vanish.

Appendix 3

In this appendix we point out briefly why the proofs of the theorems of Capper and Duff (1974) and van Nieuwenhuizen (1974) are inapplicable to graviton loops. If we rewrite the appendix of Capper and Duff (1974) in terms of our graviton field $\phi_{\mu\nu}$ defined by equation (2.2) then $\phi_{\mu\nu}$ is coupled to a conserved, traceless current $J_{\mu\nu}$†. We may then write (Bjorken and Drell 1965, Raman 1968)

$$\langle 0|TJ_{\mu\nu}(x)J_{\alpha\beta}(y)|0\rangle = \int_0^\infty d\sigma^2 \rho(\sigma^2)\mathscr{D}_{\mu\nu\alpha\beta}(\partial)\Delta_F(x-y,\sigma^2) \tag{A.51}$$

where

$$\theta(q_0)\rho(q^2)\mathscr{D}_{\mu\nu\alpha\beta}(q) \equiv (2\pi)^3 \sum_n \delta^4(p_n - q)\langle 0|J_{\mu\nu}(0)|n\rangle\langle n|J_{\alpha\beta}(0)|0\rangle \tag{A.52}$$

and, as it was argued by Capper and Duff (1974), $\rho(q^2)$ and hence the coefficient of the counterterm are both positive. This is analogous to the photon self-energy in quantum electrodynamics where, as is well known, all the infinite contributions come in with the same sign. The reason why this is true is that the photon self-energy is independent of the gauge and hence we may choose to evaluate the self-energy in a gauge (the radiation gauge) in which only the two transverse polarisation states of the photon are present. Then, if the photon couples to a current J_μ, we can write

$$\langle 0|TJ_\mu(x)J_\nu(y)|0\rangle = \int_0^\infty d\sigma^2 \rho(\sigma^2)\mathscr{D}_{\mu\nu}(\partial)\Delta_F(x-y,\sigma^2) \tag{A.53}$$

where

$$\theta(q_0)\rho(q^2)\mathscr{D}_{\mu\nu}(q) \equiv (2\pi)^3\delta^4(p_n - q)\langle 0|J_\mu(0)|n\rangle\langle| nJ_\nu(0)|0\rangle. \tag{A.54}$$

The sum over intermediate states n is only over positive norm states and ρ is indeed positive. It then follows that ρ is positive in any gauge. The same argument does not apply to the electron self-energy, which is gauge dependent.

In order to convince the reader that the problem is totally unrelated to the fictitious particles which have to be introduced into non-Abelian gauge theories, such as gravity, we consider the theory of massless scalar electrodynamics given by the Lagrangian

$$\mathscr{L} = (\partial_\mu\phi^* - ieA_\mu\phi^*)(\partial_\mu\phi + ieA_\mu\phi) - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \tag{A.55}$$

where ϕ is a complex scalar field and A_μ is the photon field. If we now choose a gauge in which the photon propagator $D_{\mu\nu}$ is given by

$$D_{\mu\nu} = \frac{1}{(2\pi)^n i(k^2 + i\epsilon)}\left(\delta_{\mu\nu} + \lambda\frac{k_\mu k_\nu}{k^2}\right), \tag{A.56}$$

then a brief calculation shows that the one-loop contribution to the scalar self-energy $\pi(\rho)$ is given by

$$\pi(p) = 60(2-5\lambda)e^2 I. \tag{A.57}$$

Clearly $\pi(p)$ is gauge dependent and the infinite part of $\pi(p)$ can change sign depending on the value of λ chosen. This is due to the fact that in general the sum over intermediate states in the spectral function analogous to equation (A.54) includes the negative norm unphysical states of the photon. Exactly the same argument applies to

† In effect we considered a theory in which only the coefficient of the spin-zero projection operator (E_4) was non-zero. This is the case for the neutrino loop considered by Capper and Duff (1974). We could equally well have made the trace of $J_{\mu\nu}$ non-zero and then the theorem would have implied that the coefficients of the E_4 and E_1 projection operators are separately positive.

the graviton self-energy where the results of Capper *et al* (1973), Capper and Namazie (1978) and Kallosh *et al* (1978) and the present paper clearly show the gauge dependence of this quantity†.

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† The photon contribution to the graviton self-energy *does* have the same sign as the scalar contribution. This is due to the fact that the graviton only couples to the photon in a gauge-invariant way (see Capper *et al* 1974, Capper 1975).